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Generic Oscillations for Delay Differential Equations

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In this article we study oscillation and nonoscillation of the solutions of linear delay differential equations. We define generic oscillation and generic nonoscillation as a topological assessment of the extent of oscillatory or nonoscillatory solutions of a delay differential equation. We provide sufficient conditions for both phenomena based on the roots of the characteristic equation and concrete conditions for some special cases. © 1998 Academic Press

Key Words: generic; oscillation; delay; roots; characteristic equation; Laplace transform.

1. INTRODUCTION

The study of oscillatory solutions of delay differential equations (DDEs) has been the focus of numerous articles (see the texts [1], [2], [3]) and their references as well as more articles ([4], [5], [6]). The strongest results deal with linear delay differential equations with constant coefficients and constant delay. In this case, necessary and sufficient conditions are given for *all* solutions to be oscillatory (see [1]). For cases of nonconstant coefficients and/or delay and for nonlinear equations, typical results give conditions sufficient for *all* solutions to be oscillatory, or conditions sufficient for *some* solutions to be nonoscillatory. In this article we examine oscillation theory in a different light. Rather than asking when all solutions are oscillatory or when there exist nonoscillatory solutions, we ask when a solution is *most likely* to be oscillatory or nonoscillatory.

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Many physical and mechanical systems are modeled by a family of delay differential equations (for instances, when hysteresis is involved, see [7]) where it is known that there are oscillatory solutions and nonoscillatory solutions. In such cases it is important to understand whether the solutions are overwhelmingly of one or the other type.

In this article we give a topological meaning to asserting that a solution is *most likely* oscillatory or nonoscillatory.

We first consider the scalar delay differential equation,

$$y'(t) = F(y(t-\tau_1), \dots, y(t-\tau_n)), \qquad (t > 0), \tag{1.1}$$

$$y(t) = \phi(t), \qquad -\tau \le t \le 0, \tag{1.2}$$

where $\tau_j \ge 0$ $(j = 1, \dots, n)$, $\tau = \max_{1 \le j \le n} \tau_j$, and $\phi \in C[-\tau, 0]$. We assume solutions of (1.1) and (1.2) exist, are unique, and extend to $[-\tau, \infty)$ for all $\phi \in C[-\tau, 0]$. We say that a solution y of (1.1) and (1.2) is oscillatory if y has arbitrarily large zeros, that is, for each M there exists $t > M$ so that $y(t) = 0$.

DEFINITION 1.1. We say that the *solutions of* (1.1) *and* (1.2) *are generically oscillatory* if the set of all $\phi \in C[-\tau, 0]$ for which the corresponding solution y_ϕ of (1.1) and (1.2) is nonoscillatory is nowhere dense in $C[-\tau, 0]$.

We say that the *solutions of* (1.1) *and* (1.2) *are generically nonoscillatory* if the set of all $\phi \in C[-\tau, 0]$ for which the corresponding solution y_ϕ of (1.1) and (1.2) is oscillatory is nowhere dense in $C[-\tau, 0]$.

A subset of $C[-\tau, 0]$ is nowhere dense in $C[-\tau, 0]$ if its closure has empty interior. As such if the solutions of (1.1) and (1.2) are generically oscillatory, then the set of initial functions that produce oscillatory solutions is overwhelming large compared to those that produce nonoscillatory solutions, and in this sense a solution is most likely oscillatory.

In Section 2, we consider the case with constant coefficients and constant delays and we present conditions based on the location of the roots of the characteristic equation that imply that the solutions are generically oscillatory or generically nonoscillatory. In Section 3, we give concrete conditions that improve well-known oscillation results. For DDEs with positive coefficients (see (2.1) in the following text) we see that if there is a nonoscillatory solution, then the solutions are generically nonoscillatory. However, a general example also reveals this is not always the case without positive coefficients.

2. ROOT CONDITIONS FOR GENERIC OSCILLATIONS FOR DELAY DIFFERENTIAL EQUATIONS

We consider the linear problem with constant coefficients and delay,

$$y'(t) + \sum_{j=1}^{n} a_j y(t - \tau_j) = 0, \qquad (t > 0), \tag{2.1}$$

$$y(t) = \phi(t), \qquad (-\tau \le t \le 0), \tag{2.2}$$

where $a_j \in R$, $\tau_j \ge 0$ $(j = 1, \dots, n)$, $\phi \in C[-\tau, 0]$, and $\tau = \max_{1 \le j \le n} \tau_j$. Existence and uniqueness of solutions of (2.1) and (2.2) are known (see [8]). The purpose of this section is to give conditions sufficient for the solutions of (2.1) and (2.2) to be generically oscillatory or to be generically nonoscillatory in terms of the roots of the characteristic equation,

$$z + \sum_{j=1}^{n} a_j e^{-z\tau_j} = 0. \tag{2.3}$$

Asymptotic stability of the trivial solution of (2.1) and oscillatory behavior of solutions of (2.1) are known (see [9], [10]). In particular, the zero solution of (2.1) is asymptotically stable if and only if all roots of (2.3) lie in the left half plane, and all solutions of (2.1) are oscillatory if and only if no roots of (2.3) are real. Criteria for generic oscillation or generic nonoscillation reduce to leading root(s) of (2.3) being real on nonreal.

DEFINITION 2.1. We say that the characteristic equation (2.3) has a *real leading root* x_0 if x_0 is a root of (2.3) and all other roots of (2.3) lie in the half plane $\{\mathrm{Re}\, z < x_0\}$. We say that (2.3) has *complex leading roots* $x_0 \pm iy_0$ $(y_0 \neq 0)$ if $x_0 \pm iy_0$ are roots of (2.3) and all other roots of (2.3) lie in the half plane $\{\mathrm{Re}\, z < x_0\}$.

THEOREM 2.1. (a) *If* (2.3) *has a real leading root, then the solutions of* (2.1) *and* (2.2) *are generically nonoscillatory.*

(b) *If* (2.3) *has complex leading roots, then the solutions of* (2.1) *and* (2.2) *are generically oscillatory.*

It is easy to see that (2.3) can have only finite many roots in any right half plane $\{\mathrm{Re}\, z > \gamma\}$, $\gamma \in R$, and thus the root conditions in Definition 2.1 are uniform. That is, in either case there is a real number $\gamma < x_0$ such that all other roots of (2.3) lie in the half plane $\{\mathrm{Re}\, z < \gamma\}$.

Proof. We employ the inversion integral for Laplace transforms. It is also known that all solutions of (2.1) are exponentially bounded (see [8]), and it is evident that the solutions are continuous on $[-\tau, \infty)$ (where it is understood that $y(0^+) = \phi(0^-)$). As such, for $t > 0$, every solution y of (2.1) and (2.2) is given by the inversion integral for the Laplace transform; that is,

$$y(t) = \frac{1}{2\pi i} \mathrm{PV} \int_{\alpha - i\infty}^{\alpha + i\infty} Y(z) e^{zt}\, dz, \tag{2.4}$$

where

$$Y(z) = \int_0^{\infty} e^{-zv} y(v)\, dv \tag{2.5}$$

is the Laplace transform of y and α is greater than the abscissa of convergence of (2.5) (see [11]). Here the integral in (2.4) is actually a principal value (as the notation PV suggests). From (2.1) and (2.2), we obtain

$$Y(z) = \frac{\phi(0) - \sum_{j=1}^{n} a_j \Psi_j(z)}{z + \sum_{j=1}^{n} a_j e^{-z\tau_j}}, \tag{2.6}$$

where

$$\Psi_j(z) = \int_0^{\tau_j} e^{-zv} \phi(v - \tau_j)\, dv. \tag{2.7}$$

Each Ψ_j is the Laplace transform of the function,

$$\psi_j(t) = \begin{cases} \phi(t - \tau_j), & \text{for } 0 \le t \le \tau_j, \\ 0, & \text{for } t > \tau_j. \end{cases}$$

Because ψ_j has a compact support, Ψ_j is an entire function.

In either statement (a) or (b) of Theorem 2.1, let x_0 be the real leading root or let $x_0 \pm iy_0$ be the complex leading roots of (2.3), respectively. Evidently, $\alpha > x_0$. However, as noted previously, we may select $\gamma < x_0$ so that all other roots of (2.3) lie in the half plane $\{\mathrm{Re}\, z < \gamma\}$, and further select $\gamma < \beta < x_0$. In either case, we apply the residue theorem to the contour shown in Fig. 1 to see that

$$\int_{C_1} Y(z) e^{zt}\, dz + \int_{C_2} Y(z) e^{zt}\, dz + \int_{C_3} Y(z) e^{zt}\, dz + \int_{C_4} Y(z) e^{zt}\, dz \tag{2.8}$$

is $2\pi i$ times the residue of $Y(z)e^{zt}$ at $z = x_0$ in statement (a) or is $2\pi i$ times the sum of the residues of $Y(z)e^{zt}$ at $z = x_0 \pm iy_0$ in statement (b). We assume that $R > |y_0|$ in the second case. In either case, $\psi_j(z)$, $e^{-z\tau_j}$, and e^{zt} are bounded on the contours C_2 and C_4 independent of R, and it

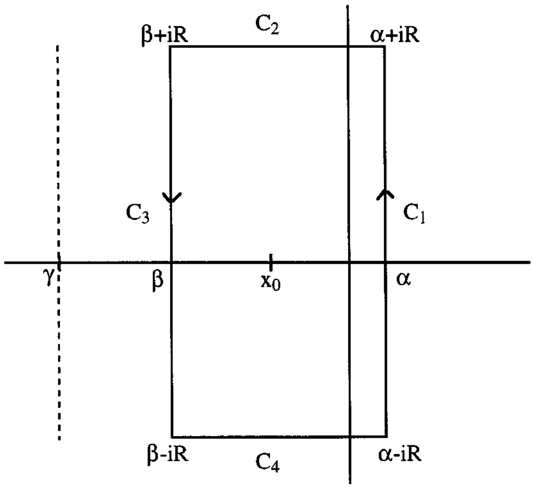

FIGURE 1

follows from (2.6) that the integrals in (2.8) over C_2 and C_4 go to 0 as $R \to \infty$.

In case (a), suppose x_0 is a root of (2.3) of order k and that

$$\phi(0) - \sum_{j=1}^{n} a_j \Psi_j(x_0) \neq 0. \tag{2.9}$$

Then

$$H(z) = \frac{(z - x_0)^k}{z + \sum_{j=1}^{n} a_j e^{-z\tau_j}}$$

is analytic at x_0 and $H(x_0) \neq 0$ (where $H(x_0)$ is suitably defined). Computing residues yield

$$y(t) = e^{x_0 t} \sum_{j=0}^{k-1} t^{k-1-j} L_j(\phi) + \frac{1}{2\pi i} \mathrm{PV} \int_{\beta - i\infty}^{\beta + i\infty} Y(z) e^{\lambda t} \, dz, \qquad (t > 0), \tag{2.10}$$

where each L_j is a continuous linear functional on $C[-\tau, 0]$, and in particular,

$$L_0(\phi) = \frac{1}{(k-1)!} H(x_0) \left(\phi(0) - \sum_{j=1}^{n} a_j \int_0^{\tau_j} e^{-x_0 v} \phi(v - \tau_j) \, dv \right)$$

is nontrivial.

In case (b), if $x_0 \pm iy_0$ are kth order roots of (2.3), then computing and summing residues yield

$$y(t) = 2e^{x_0 t} \sum_{j=0}^{k-1} t^{k-1-j} \cos\big(y_0 t + \theta_j\big) \sqrt{\big[L_j^1(\phi)\big]^2 + \big[L_j^2(\phi)\big]^2}$$
$$+ \frac{1}{2\pi i} \mathrm{PV} \int_{\beta - i\infty}^{\beta - i\infty} Y(z) e^{zt}\, dz, \qquad (t > 0), \tag{2.11}$$

for some constants θ_j and continuous linear functionals L_j^1 and L_j^2 on $C[-\tau, 0]$, $j = 0, \ldots, k-1$. The linear functional evaluations $L_0^1(\phi)$ and $L_0^2(\phi)$ are the real and imaginary parts of

$$\frac{H(x_0 + iy_0)}{(k-1)!} \left(\phi(0) - \sum_{j=1}^{n} a_j \int_0^{\tau_j} e^{-(x_0 + iy_0)v} \phi(v - \tau_j)\, dv \right),$$

respectively, and are not both trivial. H is analogous to that of case (a). It now suffices to prove that the principal values in (2.10) and (2.11) are $O(e^{\beta t})$ (and thus, $o(e^{x_0 t})$) as $t \to \infty$. In case (a), (2.10) would then imply that the solution of (2.1) and (2.2) is nonoscillatory when $L_0(\phi) \neq 0$ (i.e., (2.9) holds), and in case (b), (2.11) would imply that the solution of (2.1) and (2.2) would be oscillatory whenever $L_0^1(\phi)$ or $L_0^2(\phi)$ is nonzero. In case (a), we obtain an oscillatory solution only if $\phi \in \ker L_0$, and in case (b) we obtain a nonoscillatory solution only if $\phi \in \ker L_0^1 \cap \ker L_0^2$. Because the kernels of nontrivial linear functionals are nowhere dense, Theorem 2.1 would then follow.

To establish that the principal value in (2.10) and (2.11) are $O(e^{\beta t})$, we observe that for $R > 0$,

$$\frac{1}{e^{\beta t}} \int_{\beta - iR}^{\beta + iR} Y(z) e^{zt}\, dz = i \int_{-R}^{R} Y(\beta + iv) e^{ivt}\, dv, \tag{2.12}$$

where

$$Y(\beta + iv) = \frac{\phi(0) - \sum_{j=1}^{n} a_j \Psi_j(\beta + iv)}{\beta + iv + \sum_{j=1}^{n} a_j e^{-\tau_j(\beta + iv)}}$$
$$= \phi(0) G(v) - \sum_{j=1}^{n} a_j \Psi_j(\beta + iv) G(v), \tag{2.13}$$

where the Ψ_j are given in (2.7) and

$$G(v) = \left(\beta + iv + \sum_{j=1}^{n} a_j e^{-\tau_j(\beta + iv)} \right)^{-1}. \tag{2.14}$$

Integrating by parts yields

$$\begin{aligned}\int_{-R}^{R} \phi(0) G(v) e^{ivt}\, dv = \frac{1}{it} \phi(0) G(R) e^{iRt} - \frac{1}{it} \phi(0) G(-R) e^{-iRt} \\ - \frac{1}{it} \phi(0) \int_{-R}^{R} G'(v) e^{ivt}\, dv.\end{aligned}$$

Then from (2.14),

$$\lim_{R\to\infty} \int_{-R}^{R} \phi(0) G(v) e^{ivt}\, dv = -\frac{1}{it} \phi(0) \int_{-\infty}^{\infty} G'(v) e^{ivt}\, dv, \tag{2.15}$$

where

$$G'(v) = \frac{-i\left(1 - \sum_{j=1}^{n} a_j \tau_j e^{-\tau_j(\beta + iv)}\right)}{\left(\beta + iv + \sum_{j=1}^{n} a_j e^{-\tau_j(\beta + iv)}\right)^2}$$

is integrable on $(-\infty, \infty)$. Thus the integral in (2.15) is bounded for $t \geq 0$. (In fact, by the Riemann–Lebesgue theorem, this integral goes to 0 as $t \to \infty$). We consider the second term of (2.13), and integration by parts yields

$$\begin{aligned}\int_{-R}^{R} G(v) \Psi_j(\beta + iv) e^{ivt}\, dv = G(R) \int_{0}^{R} \Psi_j(\beta + i\rho) e^{i\rho t}\, d\rho \\ - G(-R) \int_{0}^{-R} \Psi_j(\beta + i\rho) e^{i\rho t}\, d\rho \\ - \int_{-R}^{R} G'(v) \int_{0}^{v} \Psi_j(\beta + i\rho) e^{i\rho t}\, d\rho\, dv.\end{aligned} \tag{2.16}$$

Now

$$\left|\int_0^v \Psi_j(\beta + i\rho)e^{i\rho t}\,d\rho\right| = \left|\int_0^v \left(\int_0^{\tau_j} e^{-(\beta+i\rho)s}\phi(s-\tau_j)\,ds\right)e^{i\rho t}\,d\rho\right|$$
$$= \left|\int_0^{\tau_j} \phi(s-\tau_j)e^{-s\beta}\left(\int_0^v e^{i\rho(t-s)}\,d\rho\right)ds\right|$$
$$= \left|\int_0^{\tau_j} \phi(s-\tau_j)e^{-s\beta}\frac{(e^{iv(t-s)}-1)}{i(t-s)}\,ds\right| \le w_j(t), \tag{2.17}$$

for some continuous function $w_j(t)$ that is independent of v for which

$$\lim_{t\to\infty} w_j(t) = 0. \tag{2.18}$$

By (2.14), (2.16), (2.17), and (2.18),

$$\lim_{R\to\infty}\int_{-R}^{R} \Psi_j(\beta + iv)G(v)e^{ivt}\,dv = \int_{-\infty}^{\infty} G'(v)\int_0^v \Psi_j(\beta + i\rho)e^{i\rho t}\,d\rho\,dv, \tag{2.19}$$

and by (2.17), (2.18), and the integrability of G' on $(-\infty,\infty)$, the right side of (2.19) goes to 0 as $t \to \infty$. The proof is complete.

3. SUFFICIENT CONDITIONS FOR DELAY DIFFERENTIAL EQUATIONS

In this section we give concrete conditions for the solutions of (2.1) and (2.2) to be generically oscillatory or generically nonoscillatory. Specifically, we generalize two theorems that appear in [2 pp. 38 and 47] as due to [12], and [13]. In the first of these theorems we have for a, $\tau > 0$, that all solutions of the equation,

$$y'(t) + ay(t-\tau) = 0 \tag{3.1}$$

are oscillatory if and only if $ae\tau > 1$. In particular, if $ae\tau \le 1$, then (3.1) has a nonoscillatory solution. In the second of these theorems we have that when $a_j, \tau_j > 0$ $(j = 1, \dots, n)$, all solutions of (2.1) are oscillatory if and only if

$$-\lambda + \sum_{j=1}^{n} a_j e^{\lambda \tau_j} > 0, \tag{3.2}$$

for all $\lambda > 0$. When $a > 0$, $\tau > 0$,

$$\min\{-\lambda + ae^{\lambda\tau} : \lambda \in R\} = \frac{1}{\tau}\left(1 - \ln \frac{1}{a\tau}\right),$$

and condition (3.2) is then easily seen to be equivalent to $ae\tau > 1$ when $n = 1$. Essentially our results show that in these cases when there is a nonoscillatory solution, the solutions are generically nonoscillatory. Just as well, we remove the requirement that $a > 0$ in the case of one delay and we relax the corresponding requirement in the case of several delays.

THEOREM 3.1. *Let $\tau \geq 0$ and $a \in R$.*

(i) *If $ae\tau > 1$, then all solutions of* (3.1) *are oscillatory.*

(ii) *If $ae\tau \leq 1$, then the solutions of* (3.1) *are generically nonoscillatory.*

THEOREM 3.2. *Let $\tau_j \geq 0$ $(j = 1, \dots, n)$*

(i) *Suppose that $a_j > 0$ $(j = 1, \dots, n)$ and* (3.2) *holds. Then all solutions of* (2.1) *are oscillatory*

(ii) *Suppose that $a_j \in R$ $(j = 1, \dots, n)$ and*

$$-\lambda_0 + \sum_{j=1}^{n} |a_j| e^{\lambda_0 \tau_j} < 0, \tag{3.3}$$

for some $\lambda_0 > 0$. Then the solutions of (2.1) *and* (2.2) *are generically nonoscillatory.*

(iii) *Suppose that $a_j > 0$ $(j = 1, \dots, n)$ and some $\tau_i > 0$. If*

$$-\lambda_0 + \sum_{i=1}^{n} a_i e^{\lambda_0 \tau_i} = 0, \tag{3.4}$$

for some $\lambda_0 > 0$ and

$$-\lambda + \sum_{j=1}^{n} a_j e^{\lambda \tau_j} \geq 0, \tag{3.5}$$

for all $\lambda > 0$, then the solutions of (2.1) *and* (2.2) *are generically nonoscillatory.*

Of course parts (i) of both theorems are not new. We include them in the statement for completeness. We prove Theorem 3.2 first and we use part of it is the proof of Theorem 3.1.

Proof of Theorem 3.2. For (i), observe that if each $a_j > 0$, then

$$x + \sum_{j=1}^{n} a_j e^{-x\tau_j} > 0, \tag{3.6}$$

for all $x > 0$. Just as well, the validity of (3.6) for all $\lambda > 0$ implies that (3.2) holds for all $x < 0$, and hence, the characteristic equation (2.3) has no real roots. By [1 p. 33], all solutions of (2.1) are oscillatory.

For (ii), select $\lambda_0 > 0$ so that strict inequality in (3.3) holds. Thus if $\operatorname{Re} z = -\lambda_0$, then

$$|z| \geq \lambda_0 > \sum_{j=1}^{n} |a_j| e^{\lambda_0 \tau_j} \geq \left| \sum_{j=1}^{n} a_j e^{-z\tau_j} \right|.$$

Now $\sum_{j=1}^{n} a_j e^{-z\tau_j}$ is bounded on the half plane $\{\operatorname{Re} z > -\lambda_0\}$, and thus there is an $R > 0$ so that $|z| > |\sum_{j=1}^{n} a_j e^{-z\tau_j}|$ on the contour D_R shown in Fig. 2 and on the intersection of the half plane $\{\operatorname{Re} z > -\lambda_0\}$ and the exterior region for the contour $C_R + D_R$ shown in Fig. 2.

It follows from Rouche's theorem that z and $z + \sum_{j=1}^{n} a_j e^{-z\tau_j}$ have the same number of roots in the interior region of $C_R + D_R$. Then in fact these functions have the same number of roots in the half plane $\{\operatorname{Re} z \geq -\lambda_0\}$. But z has precisely one root there so that the characteristic

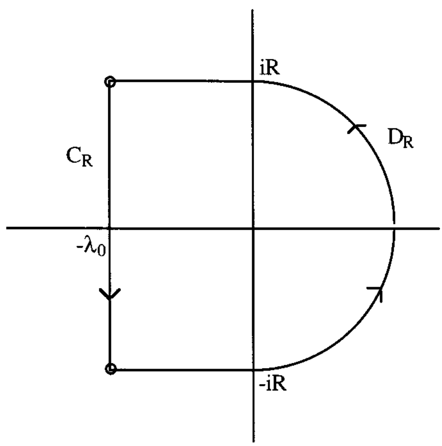

FIGURE 2

equation (2.3) has precisely one root in $\{\operatorname{Re} z \geq \lambda_0\}$. It thus follows that the leading root of (2.3) is real, and by Theorem 2.1 the solutions of (2.1) and (2.2), are generically nonoscillatory. Theorem 3.2(ii) is now proven.

For (iii), let

$$w = z + \sum_{j=1}^{n} a_j e^{-z\tau_j}. \tag{3.7}$$

Then

$$w'' = \sum_{j=1}^{n} \tau_j^2 a_j e^{-z\tau_j} > 0, \tag{3.8}$$

when z is real, and by Rolle's theorem and the fact that $w' = 0$ for $z = -\lambda_0$, $-\lambda_0$ is the only real root of Eq. (3.4). We claim that w has no complex roots in $\{\operatorname{Re} z > -\lambda_0\}$. Suppose not. Let z_1 be a complex root in $\{\operatorname{Re} z > -\lambda_0\}$. Because w is analytic, there is a circle C about z_1 disjoint from real axis and $\{\operatorname{Re} z = -\lambda_0\}$ on which w has no zeros. Choose $0 < \varepsilon < a_1$ such that $|w| > \varepsilon|e^{-z\tau_1}|$ on C. Thus implies by Rouche's theorem, that w and $w - \varepsilon e^{-z\tau_1}$ have the same number of roots in the interior region of the circle C. Thus $w - \varepsilon e^{-z\tau_1}$ has a complex root in $\{\operatorname{Re} z > -\lambda_0\}$. But

$$-\lambda_0 + (a_1 - \varepsilon)e^{\lambda_0\tau_1} + \sum_{j=2}^{n} a_j e^{\lambda_0\tau_1} = w(-\lambda_0) - \varepsilon e^{\lambda_0\tau_1} = -\varepsilon e^{\lambda_0\tau_1} < 0. \tag{3.9}$$

By the proof of part (ii) $w - \varepsilon e^{z\tau_1}$ has exactly one zero in $\{\operatorname{Re} z > -\lambda_0\}$ and it is real, a contradiction. Thus, w has no complex roots in $\{\operatorname{Re} z > -\lambda_0\}$. Consider z where $\operatorname{Re} z = -\lambda_0$ but z is not real. Then

$$|z| > -\lambda_0 = \sum_{j=1}^{n} a_j e^{\lambda_0\tau} \geq \sum_{j=1}^{n} a_j|e^{-z\tau_j}| \geq \left| \sum_{j=1}^{n} a_j e^{-z\tau_j} \right|, \tag{3.10}$$

and $w(z) \neq 0$. Therefore $-\lambda_0$ is the leading root of w, which implies that the solutions of (2.1) and (2.2) are generically nonoscillatory.

Proof of Theorem 3.2. In view of Theorem 3.2, we only need to show that when $a \leq 0$, the solutions of (3.1) are generically nonoscillatory. When $a = 0$, this result is clear. So we consider $a < 0$. If $a < 0$, there is a unique real simple solution x_0 to $x_0 = -ae^{-\tau x_0}$ and $x_0 > 0$. For $z = x + iy$ ($y \neq 0$) with $x \geq x_0$, we have $|z| = \sqrt{x^2 + y^2} > x_0$ and $|-ae^{-\tau_0 z}| = -ae^{-\tau_0 x} \leq -ae^{-\tau_0 x_0} = x_0 < |z|$. Thus x_0 is the real leading root of (2.3), and Theorem 3.1 is proven.

Remark. In the case of one delay, Theorem 3.1 yields that if there is a nonoscillatory solution of (2.1) and (2.2), then the solutions are generically nonoscillatory. In the case of several delays and positive coefficients, Theorem 3.2 provides the same. One might ask whether it is at all possible for the solutions of (2.1) and (2.2) to be generically oscillatory while some solutions are nonoscillatory. In other words can we have a nontrivial generically oscillatory solutions of (2.1) and (2.2)? In the next general example we answer this question in the affirmative.

EXAMPLE. Select $a_j > 0$ and $\tau_j > 0$ $(j = 1, \dots, n)$ so that

$$x + \sum_{j=1}^{n} a_j e^{-x\tau_j} > 0, \tag{3.11}$$

for all $x < 0$. Assume the leading root of (2.3) has multiplicity 1. From inequality (3.11) all solutions of (2.1) and (2.2) are oscillatory and the function,

$$F(z) = z + \sum_{j=1}^{n} a_j e^{-z\tau_j}$$

has two complex leading roots. Select $\tau = \max_{1 \le j \le n} \tau_j$, and let

$$w(z) = z + \sum_{j=1}^{n} a_j e^{-z\tau_j} - \varepsilon e^{-z\sigma}, \tag{3.12}$$

where $\varepsilon > 0$ and $\sigma > \tau$.

Choose v so that $\{\operatorname{Re} z > v\}$ contains the leadings roots of $F(z)$, but that $\{\operatorname{Re} z \ge v\}$ contains no other roots of $F(z)$. Using circles about these roots disjoint from $\{\operatorname{Re} z = v\}$ and the real axis, an application of Rouche's theorem yields that w has two nonreal roots in $\{\operatorname{Re} z > v\}$ for ε sufficiently small.

Evidently,

$$\sum_{j=1}^{n} |a_j e^{-z\tau_j}| + |e^{-z\sigma}| \tag{3.13}$$

is bounded on $\{\operatorname{Re} z \ge v\}$. Choose R such that

$$|z| > \sum_{j=1}^{n} |a_j e^{-z\tau_j}| + |e^{-z\sigma}|, \tag{3.14}$$

when $|z| \ge R$ and $\operatorname{Re} z \ge v$.

It follows that

$$\left| z - \sum_{j=1}^{n} a_j e^{-z\tau_j} \right| \geq |z| - \sum_{j=1}^{n} |a_j e^{-z\tau_j}| > |e^{-z\sigma}| > |\varepsilon e^{-z\sigma}|, \quad (3.15)$$

for $0 < \varepsilon < 1$ when $|z| \geq R$ and $\operatorname{Re} z \geq \upsilon$. Also by the choice of υ,

$$\upsilon + \sum_{j=1}^{n} a_j e^{-\upsilon\tau_j}$$

is nonzero on C_R. The contours D_R and C_R are the portion of the circle of radius R about the origin lying in $\{\operatorname{Re} z \geq \upsilon\}$ and the portion of the line $\operatorname{Re} z = \upsilon$ lying in the disk of radius R about the origin. Now for ε sufficiently small,

$$\left| z + \sum_{j=1}^{n} a_j e^{-z\tau_j} \right| > |\varepsilon e^{-z\sigma}|, \quad (3.16)$$

for z on $D_R + C_R$, and by Rouche's theorem w and $z + \sum_{j=1}^{n} a_j e^{-z\tau_j}$ have exactly two roots inside $D_R + C_R$. Also note that w has no other roots in $\{\operatorname{Re} z \geq \upsilon\}$ and therefore the roots are complex. This implies that w has complex leading roots which implies the solutions of

$$y'(t) + \sum_{j=1}^{n} a_j y(t - \tau_j) - \varepsilon y(t - \sigma) = 0 \quad (3.17)$$

are generically oscillatory. On the other hand,

$$\lim_{\substack{x \to \infty \\ x \text{ real}}} w = \infty, \quad (3.18)$$

and because $\sigma > \tau$,

$$\lim_{\substack{x \to -\infty \\ x \text{ real}}} w = -\infty. \quad (3.19)$$

Equations (3.18) and (3.19) imply that w has a real root and therefore in this case Eq. (3.17) has a nonoscillatory solution.

Remark 1. The results of this article are for first-order linear DDEs with constant coefficients. This is the first study on generic oscillations. Generic oscillation can yield more insight into solutions of DDEs than the standard osciallation theorems, and as such, it is important to further study this topic for nonlinear DDEs and for systems of DDEs. The authors believe that different techniques need to be developed.

Remark 2. This remark deals with the existence of leading roots. However, as noted after the statement of Theorem 2.1, there do exist "right-most" roots of (2.3) in the complex plane. This article does not apply to the two seemingly rare cases when there is a tie for the right-most roots between a real root and a complex pair or between two or more complex pairs. It can be shown that when the coefficients $a_1, \ldots, a_n$ in (2.1) are of the same sign, then there cannot be a tie for the right-most roots between a real root and a complex pair. The authors consider the following to be a very difficult open question. If there is a tie for right-most roots between a real root and a complex pair, is it possible for neither of the sets of initial functions leading to oscillatory or nonoscillatory solutions to be nowhere dense in $C[-\tau, 0]$? In the case of a tie among complex pairs, it can be shown that the conclusion of Theorem 2.1(b) holds if the imaginary parts of right-most roots represent at most two commensurability classes. The authors believe that this conclusion holds even without the commensurability class condition.

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